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Chiral Actions and Einstein's Vacuum Equations

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The Euler–Lagrange equations of recently introduced chiral action principles are discussed using Lie algebra-valued differential forms. Symmetries of the equations and the chiral description of Einstein's vacuum equations are presented. A class of Lagrangians which contains the chiral formulations is exhibited.

1. INTRODUCTION

Recently a complex chiral action was presented in which the field variables are an $sl(2, \mathbb{C})$-valued connection 1-form, two (2-component) spinor-valued 1-forms, and two spinor-valued 2-forms (Robinson, 1996). When the 1-forms are linearly independent the classical theory so defined corresponds to vacuum general relativity in four dimensions. The 1-forms define a (complex) 4-metric and the field equations imply that the $sl(2, \mathbb{C})$ connection 1-form is the anti-self-dual (Levi-Civita) spin connection. The Ricci tensor of this connection is zero. Real general relativity can be recovered from the complex theory by the imposition of reality conditions. When the 1-forms are not linearly independent the field equations define a generalization of Einstein's vacuum field equations which is determined by a degenerate 4-metric and a connection. Degenerate metrics have been of recent interest in quantum gravity.

Since the spinor-valued 1-forms define spin-3/2 fields, an alternative approach to the formalism is to treat them as Grassman-valued (anticommuting) fields as in supergravity. Aspects of this latter approach are contained in earlier work (Bars and MacDowell, 1977, 1979), where the pair of 1-forms are determined by an anticommuting Majorana spin-3/2 field.

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The principal aim of this paper is to investigate the Lie algebras related to the Euler–Lagrange equations of the complex chiral action and the related action of Tung and Jacobson (1995; Tung, 1996). In these Lagrangian formulations the spinor-valued forms and the connection 1-form are treated as classical (c-number) fields. They are regarded as the primary field variables rather than the metrics which can exist as secondary, composite objects. Investigation of the related Lie algebras shows that these formalisms can be viewed as belonging to a wider class of Lagrangian field theories. Members of this class are distinguished from one another by their gauge groups.

In Section 2 the chiral Lagrangian formalism is briefly reviewed and the case where a regular 4-metric is defined is briefly discussed. Two standard formulations of Cartan's structure equations are recalled in Section 3, first in terms of differential forms which take their values in the Lie algebra of g, the (affine) group of motions of a (pseudo-) Riemannian metric in n dimensions, and second in terms of two-component spinors in four dimensions. These formulations provide the background for the following discussion of the chiral Lagrangians and field equations. The Lie algebras naturally associated with the latter are identified and it is shown that the field equations can be expressed as one simple equation for Lie algebra-valued forms. In Section 4 it is demonstrated that these results can be extended naturally to Lagrangian field theories with other gauge groups. In Section 5, certain Lagrangians with two connections as field variables are considered and shown to be equivalent to the (generalized) Tung–Jacobson Lagrangians. Finally a chiral four-spinor notation is introduced and used to describe Lagrangian formulations of Einstein's vacuum equations.

Lowercase Latin indices label Lie algebra generators and differential forms. Uppercase Latin indices range and sum over 0 and 1, and standard two-component spinor conventions are followed. Spinor indices are raised and lowered with the constant symplectic spinors ε^{AB} and ε_{AB} (Penrose and Rindler, 1984).

2. THE CHIRAL LAGRANGIANS AND FIELD EQUATIONS

The Lagrangian 4-form, (Robinson, 1996) is

$$L = \nu_A \wedge D\alpha^A + D\beta^A \wedge \mu_A - \nu_A \wedge \mu^A \tag{1}$$

where α^A and β^A are spinor-valued 1-forms, μ^A and ν^A are spinor-valued 2-forms, and D is the covariant exterior derivative of an $sl(2, \mathbb{C})$-valued connection 1-form Γ^A_B. The Euler–Lagrange equations corresponding to variations with respect to the 2-forms ν_A and μ_A, the 1-forms α_A and β_A, and the connection Γ^A_B are

$$D\alpha^A - \mu^A = 0, \qquad D\beta^A + \nu^A = 0 \tag{2}$$

$$D\nu^A = 0, \qquad D\mu^A = 0 \tag{3}$$

and

$$\nu^{(A} \wedge \alpha^{B)} + \beta^{(A} \wedge \mu^{B)} = 0 \tag{4}$$

The Lagrangian of Tung and Jacobson (1995),

$$L_1 = D\beta^A \wedge D\alpha_A \tag{5}$$

is obtained from L when equations (2) are satisfied.

When α^A and β^B are linearly independent (the regularity condition) they determine the metric

$$ds^2 = \alpha_A \otimes \beta^A + \beta^A \otimes \alpha_A \tag{6}$$

When α^A and β^A constitute a coframe, equation (4) is satisfied if and only if there exist unique 1-forms π_1, π_2, π_3 such that

$$\begin{aligned} \mu^A &= \alpha^A \wedge \pi_1 + \beta^A \wedge \pi_2 \\ \nu^A &= -\alpha^A \wedge \pi_3 + \beta^A \wedge \pi_1 \end{aligned} \tag{7}$$

Then the remaining field equations (3) are satisfied if and only if

$$\alpha^A \wedge D\pi_1 + \beta^A \wedge D\pi_2 + 2\beta^A \wedge \pi_1 \wedge \pi_2 + \alpha^A \wedge \pi_2 \wedge \pi_3 = 0 \tag{8}$$

and

$$\alpha^A \wedge D\pi_3 - \beta^A \wedge D\pi_1 - 2\alpha^A \wedge \pi_1 \wedge \pi_3 - \beta^A \wedge \pi_2 \wedge \pi_3 = 0 \tag{9}$$

When the regularity condition is satisfied these chiral equations are a formulation of Einstein's vacuum equations for a metric. Equations (2) and (4) imply that Γ^A_B is the anti-self-dual part of the Levi-Civita (spin) connection. Equations (3) are equivalent to Einstein's vacuum field equations. The three 1-forms π_1, π_2, and π_3 represent the components $\Gamma_{A'B'}o^{A'}\iota^{B'}$, $\Gamma_{A'B'}\iota^{A'}\iota^{B'}$, and $-\Gamma_{A'B'}o^{A'}o^{B'}$ of the self-dual part of the Levi-Civita (spin) connection, and the spinor fields

$$\zeta_{A'} = -\pi_1\iota_{A'} + \pi_2 o_{A'} \qquad \text{and} \qquad \xi_{A'} = -\pi_3\iota_{A'} - \pi_1 o_{A'} \tag{10}$$

satisfy the Rarita–Schwinger spin-3/2 zero-rest-mass field equations.

When the regularity condition is not satisfied, solutions of the field equations may be interpreted as defining gravitational vacuum equations for degenerate metrics. The latter are of interest in the context of quantum gravity.

3. CARTAN'S AND EINSTEIN'S EQUATIONS AND LIE ALGEBRAS

Einstein's vacuum field equations and Cartan's structure equations for a metric geometry can be formulated in terms of Lie algebra-valued forms as follows. Let $\theta = \theta^a P_a$ and $\Gamma = \frac{1}{2}\Gamma^a_b J^b_a$ be Lie algebra-valued 1-forms, where θ^a and Γ^a_b are, respectively, the components of an orthonormal coframe for the metric

$$ds^2 = g_{ab}\theta^a \otimes \theta^b \tag{11}$$

and the components of a connection one-form. The generators of the affine Lie algebra g satisfy the commutation relations

$$[P_a, P_b] = 0; \qquad [P_a, J_{bc}] = (P_b g_{ac} - P_c g_{ab})$$
$$[J_{ab}, J_{cd}] = (g_{bc}J_{da} + g_{da}J_{cb} + g_{bd}J_{ac} + g_{ac}J_{bd}) \tag{12}$$

where $J_{ab} = -J_{ba}$.

Then Cartan's first structure equations are

$$\Theta = d\theta + [\theta, \Gamma] \tag{13}$$

where $\Theta = \Theta^a P_a$ is the torsion 2-form. Cartan's second structure equations are

$$F = d\Gamma + \frac{1}{2}[\Gamma, \Gamma] \tag{14}$$

and $F = \frac{1}{2}F^a_b J^b_a$ is the curvature 2-form.

In the special case of four-dimensional manifolds, to which henceforth this paper will be restricted, the Einstein vacuum field equations may be written as

$$\Theta = 0; \qquad [\theta, *F] = 0 \tag{15}$$

where $*F$ is the left dual, $\frac{1}{4}\varepsilon^{ad}_{bc} F^c_d J^b_a$, of F.

When $n = 4$ the Lie algebra equations (12) can be reexpressed in terms of two-component spinors as

$$[P_{AA'}, P_{BB'}] = 0; \qquad [P_{AA'}, J_{BC}] = -\frac{1}{2}\varepsilon_{AC} P_{BA'} - \frac{1}{2}\varepsilon_{AB} P_{CA'} \tag{16a}$$

$$[J_{AB}, J_{CD}] = \frac{1}{2}(\varepsilon_{CB} J_{AD} + \varepsilon_{DA} J_{BC} + \varepsilon_{CA} J_{DB} + \varepsilon_{DB} J_{CA}) \tag{16b}$$

$$[J_{AB}, J_{A'B'}] = 0 \tag{16c}$$

with similar equations for $[P_{AA'}, J_{B'C'}]$ and $[J_{A'B'}, J_{C'D'}]$. Here $P_{AA'} \leftrightarrow P_a$ and $\delta^A_B J^{A'}_{B'} + \delta^{A'}_{B'} J^A_B \leftrightarrow J^a_b$, with $J_{AB} = J_{BA}$ and $J_{A'B'} = J_{B'A'}$. The spinor notation exhibits explicitly the $so(4, \mathbb{C})$ Lie algebra isomorphism $so(4, \mathbb{C}) \simeq sl(2, \mathbb{C}) \oplus sl(2, \mathbb{C})$, or, with the appropriate reality conditions, $so(4, \mathbb{R}) \simeq su(2) \oplus su(2)$, $so(1, 3) \simeq sl(2, \mathbb{C})$; $so(2, 2) \simeq su(1, 1) \oplus su(1, 1) = sl(2, \mathbb{R}) \oplus sl(2, \mathbb{R})$. These isomorphisms imply that the second Cartan structure equations can be written in terms of the self-dual and anti-self-dual parts of the connection $\overset{(+)}{\Gamma} = \Gamma^{A'}_{B'} J^{B'}_{A'}$ and $\overset{(-)}{\Gamma} = \Gamma^{A}_{B} J^{B}_{A}$ where $\Gamma = \overset{(+)}{\Gamma} + \overset{(-)}{\Gamma}$ and $\Gamma^a_b \leftrightarrow \delta^{A'}_{B'} \Gamma^A_B + \delta^A_B \Gamma^{A'}_{B'}$. Then

$$\overset{(+)}{F} = d\overset{(+)}{\Gamma} + \frac{1}{2}[\overset{(+)}{\Gamma}, \overset{(+)}{\Gamma}]; \qquad \overset{(-)}{F} = d\overset{(-)}{\Gamma} + \frac{1}{2}[\overset{(-)}{\Gamma}, \overset{(-)}{\Gamma}] \tag{17}$$

where $F = \overset{(+)}{F} + \overset{(-)}{F}$ and $\overset{(-)}{F} = F^A_B J^B_A$, $\overset{(+)}{F} = F^{A'}_{B'} J^{B'}_{A'}$.

The Einstein vacuum equations in four dimensions can now be written as

$$\Theta = 0, \qquad [\overset{(+)}{F}, \theta] = 0 \tag{18}$$

or, alternatively,

$$\Theta = 0, \qquad [\overset{(-)}{F}, \theta] = 0 \tag{19}$$

The chiral equations (2)–(4), are not *a priori* metric equations, but the relevant Lie algebras and Lie algebra-valued 1-forms can be related to the above equations as follows.

Let $(o^{A'}, \iota^{A'})$ be a constant spin dyad with $o_{A'}\iota^{A'} = 1$. By defining the Lie algebra generators

$$a_A = P_{AA'}o^{A'}, \qquad b_A = P_{AA'}\iota^{A'} \tag{20}$$

we can write equations (16a) as

$$[a_A, a_B] = 0; \qquad [a_A, J_{BC}] = -\frac{1}{2}\varepsilon_{AC}a_B - \frac{1}{2}\varepsilon_{AB}a_C \tag{21a}$$

$$[a_A, b_B] = 0 \tag{21b}$$

and

$$[b_A, b_B] = 0; \qquad [b_A, J_{BC}] = -\frac{1}{2}\varepsilon_{AC}b_B - \frac{1}{2}\varepsilon_{AB}b_C \tag{21c}$$

Equations (21) and (16b) specify a chiral representation of a semi-direct sum of $sl(2, \mathbb{C})$ and $\mathbb{C}^4$ or, when reality conditions apply, a semi-direct sum of subalgebras.

The Euler–Lagrange equations (2)–(4) can be expressed in terms of one-forms with values in this (semi-direct sum) Lie algebra. Define the Lie algebra-valued one-forms

$$\theta = \alpha^A a_A + \beta^A b_A \tag{22}$$

and, as before, $\overset{(-)}{\Gamma} = \Gamma^A_B J^B_A$. The first Cartan equation corresponding to the chiral algebra is

$$\overset{(-)}{\Theta} = d\theta + [\theta, \overset{(-)}{\Gamma}] \tag{23}$$

and defines a complex torsionlike two-form of the connection $\overset{(-)}{\Gamma}$. Direct calculation shows that equation (23) is equivalent to equation (2) with

$$\overset{(-)}{\Theta} = \mu^A a_A - \nu^A b_A \tag{24}$$

If D is the covariant exterior derivative determined by $\overset{(-)}{\Gamma}$, then equations (3) and (4) are respectively equivalent to the equations

$$D\overset{(-)}{\Theta} = d\overset{(-)}{\Theta} - [\overset{(-)}{\Theta}, \overset{(-)}{\Gamma}] = 0 \tag{25}$$

and

$$D\overset{(-)}{\Sigma} = 0 \tag{26}$$

where

$$\overset{(-)}{\Sigma} = \alpha^B \wedge \beta^A J_{AB} \tag{27}$$

The second Cartan equations are as in equation (17). With $\overset{(-)}{\Theta}$ and $\overset{(-)}{\Sigma}$ defined as above, the Einstein vacuum equations given by equations (3) and (4) can be summarized in the single concise equation

$$D(\overset{(-)}{\Theta} + \overset{(-)}{\Sigma}) = 0 \tag{28}$$

When reality conditions are imposed the equations corresponding to the real 4-dimensional geometries are obtained. By introducing the conjugation operation

$$a_A \mapsto a^\dagger_A = \bar{a}_{B'} t^{B'}_A; \qquad t^{B'}_A = \begin{bmatrix} 0 & 1 \\ -\sigma & 0 \end{bmatrix} \tag{29}$$

and dually

$$\alpha^A \mapsto \alpha^{\dagger A} = T^A_{B'}\overline{\alpha}^{B'}; \qquad T^A_{B'} = \begin{bmatrix} 0 & -\sigma \\ 1 & 0 \end{bmatrix} \tag{30}$$

where $\sigma = +1$ for real geometries with Euclidean signature and $\sigma = -1$ for real geometries with ultrahyperbolic signature (Mason and Woodhouse, 1996), the choices

$$\beta^A = \alpha^{\dagger A} \qquad \text{and} \qquad \nu^A = -\,\mu^{\dagger A} \tag{31}$$

can be made for these geometries.

Then the Lagrangian given in equation (1) becomes

$$L = -\mu^\dagger_A \wedge D\alpha^A + D\alpha^{\dagger A} \wedge \mu_A + \mu^\dagger_A \wedge \mu^A \tag{32}$$

and the Euler–Lagrange equations reduce to the system

$$D\alpha^A - \mu^A = 0 \tag{33a}$$

$$D\mu^A = 0 \tag{33b}$$

and

$$-\mu^{\dagger(A} \wedge \alpha^{B)} + \alpha^{\dagger(A} \wedge \mu^{B)} = 0 \tag{33c}$$

The Tung–Jacobson Lagrangian given in equation (5) reduces to $L_1 = D\alpha^{\dagger A} \wedge D\alpha_A$.

Now the relevant Lie algebra is the semi-direct sum of $su(2)$ [respectively $su(1, 1)$] and $\mathbb{C}^2$ with commutation relations given by equations (21a) and (16b). The corresponding Cartan equations can be expressed in terms of $\overset{(-)}{\Gamma}$ and the Lie algebra-valued 1-form $\overset{(-)}{\theta} = \alpha^A\sigma_A$. Equation (33a) is the first Cartan equation corresponding to this algebra, that is,

$$\underset{1}{\overset{(-)}{\Theta}} = d\overset{(-)}{\theta} + [\overset{(-)}{\theta}, \overset{(-)}{\Gamma}] \tag{34}$$

with torsionlike two-form

$$\underset{1}{\overset{(-)}{\Theta}} = \mu^A a_A$$

The Einstein vacuum equations for these geometries, equations (33b) and (33c), can now be written as

$$D(\underset{1}{\overset{(-)}{\Theta}} + \overset{(-)}{\Sigma}) = 0 \tag{35}$$

In the case of Lorentzian geometries the reality conditions can be imposed by requiring that $\alpha^0 = \overline{\alpha}^0$, $\beta^1 = \overline{\beta}^1$, $\alpha^1 = \overline{\beta}^0$, $a_0 = \overline{a}_0$, $b_1 = \overline{b}_1$, $a_1 = \overline{b}_0$, and the underlying Lie algebra is the semi-direct sum of $sl(2, \mathbb{C})$ and $\mathbb{R}^4$ as in equations (21) and (16b).

Further symmetries of the field equations (2)–(4) are given by the transformations

$$\begin{aligned} \alpha^A &\mapsto p\alpha^A + q\beta^A;\ \beta^A \mapsto r\alpha^A + s\beta^A \\ \mu^A &\mapsto Dp \wedge \alpha^A + Dq \wedge \beta^A + p\mu^A - q\nu^A \\ \nu^A &\mapsto -Dr \wedge \alpha^A - Ds \wedge \beta^A - r\mu^A + s\nu^A \end{aligned} \tag{36}$$

where $ps - qr$ is a nonzero constant k. Equation (36) is an anti-self-dual representation of the standard self-dual $sl(2, \mathbb{C})$ (when $k = 1$) gauge transformations (Tung, 1996). Under these transformations the chiral Lagrangian given in equation (1) transforms as

$$L \mapsto kL + dE \tag{37}$$

where

$$\begin{aligned} 2E = {} & (rDp - pDr) \wedge \alpha_A \wedge \alpha^A + (sDq - qDs) \wedge \beta_A \wedge \beta^A \\ & + (rDq - qDr + sDp - pDs) \wedge \alpha_A \wedge \beta^A \end{aligned} \tag{38}$$

4. A LARGER CLASS OF LAGRANGIANS AND LIE ALGEBRAS

Vacuum general relativity, formulated as a Lagrangian field theory as in the previous sections, can be regarded as one member of a class of field theories. Members of this class differ from general relativity by having gauge groups other than $SL(2, \mathbb{C})$. This will be illustrated here by considering Lagrangians with gauge group $\mathcal{G}$, where $\mathcal{G}$ is either $SO(\mathrm{N}, \mathbb{C})$ or $Sp(N, \mathbb{C})$. However, as is clear from the previous section, the class could be extended to include theories with unitary and other gauge groups. Consider the Lagrangian

$$L_2 = \nu^j \wedge D\alpha^i h_{ji} + D\beta^i \wedge \mu^j h_{ji} - \nu^i \wedge \mu^j h_{ij} \tag{39}$$

where α^i and β^i are 1-forms, ν^i and μ^i are 2-forms, all with values in (a matrix representation of) the Lie algebra of $\mathcal{G}$. The covariant exterior derivative D corresponds to a Lie algebra-valued connection 1-form Γ^i_j and $\mathcal{G}$ is the isometry group of the (constant and covariantly constant) metric h_{ij}, so that, for all $\mathrm{M}^i_j \in \mathcal{G}$,

$$M^i_j M^k_l h_{ik} = h_{jl} \tag{40}$$

and

$$Dh_{ij} = -h_{kj}\Gamma^k_i - h_{ik}\Gamma^k_j = 0 \tag{41}$$

Variation of the Lagrangian L_2 with respect to the field variables leads to the equations

$$D\alpha^i - \mu^i = 0; \qquad D\beta^i h_{ji} - \nu^i h_{ij} = 0, \tag{42}$$

$$D\mu^i = 0; \qquad D\nu^i = 0 \tag{43}$$

and

$$\tau_{ij}(\beta^i \wedge \mu^j + \nu^j \wedge \alpha^i) = 0 \tag{44}$$

Here τ_{ij} is an arbitrary field with the symmetries of $h_{ki}\delta\Gamma^k_j$.

When equations (39) are satisfied L_2 reduces to

$$L_3 = h_{ij} D\alpha^i \wedge D\beta^j \tag{45}$$

The Euler–Lagrange equations determined by L_3 are equations (43) and, when either $h_{ij} = h_{ji}$ or $h_{ij} = -h_{ji}$ and h_{ij} is nondegenerate, equation (44).

The Lie algebra of $\mathcal{G}$ with commutators

$$[J_{ij}, J_{kl}] = \frac{1}{2}(h_{kj}J_{il} + h_{li}J_{jk} + h_{ki}J_{lj} + h_{lj}J_{ki}) \tag{46}$$

can be extended to a semi-direct sum with two Abelian Lie algebras whose generators are, respectively, a_i and b_i. The semi-direct sum is defined by the commutators

$$[a_i, J_{jk}] = -\frac{1}{2}h_{ik}a_j - \frac{1}{2}\sigma h_{ij}a_k \tag{47a}$$

and

$$[b_i, J_{jk}] = -\frac{1}{2}h_{ik}b_j - \frac{1}{2}\sigma h_{ij}b_k \tag{47b}$$

where $\sigma = +1$ when $h_{ij} = -h_{ji}$ and $\sigma = -1$ when $h_{ij} = h_{ji}$.

Introducing Lie algebra-valued forms defined as $\theta = \alpha^i a_i + \beta^i b_i$, $\Gamma = \sigma\Gamma^i_j J^j_i$, $\Theta = d\theta + [\theta, \Gamma] = \mu^i a_i - \nu^i b_i$, and $\Sigma = \alpha^i \wedge \beta^j J_{ij}$ enables the field equations (43)–(44) to be written in the same concise form as Einstein's vacuum equations, that is,

$$D(\Theta + \Sigma) = 0 \tag{48}$$

5. DISCUSSION

It has been shown that Einstein's vacuum field equations can be given Lagrangian formulations which place them naturally in a class of more general gauge theories. The semi-direct sum structures of the Lie algebras considered above suggest the consideration of two connection 1-forms A_1 and A_2 with matrix representations

$$A_1 = \begin{pmatrix} \Gamma^i_j & \alpha^i \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad A_2 = \begin{pmatrix} \Gamma^i_j & 0 \\ \beta^i h_{ji} & 0 \end{pmatrix} \tag{49}$$

and curvatures F_1 and F_2, respectively. Then the Lagrangian 4-form $Tr(F_1 \wedge F_2)$ is equal to $D\alpha^i \wedge D\beta^j h_{ij} + F^i_j \wedge F^j_i$. This differs from the generalized Tung–Jacobson Lagrangian (45) by the exterior derivative of the Chern–Simons 3-form. Thus this Lagrangian, which depends only on two connections, gives the same field equations as L_3 (and L_2). In the special case where the gauge group is $SL(2, \mathbb{C})$, the Einstein vacuum equations are therefore formulated in terms of a two-connection Lagrangian (Barbero, 1994).

The Lagrangians for Einstein's vacuum equations presented in earlier sections use the anti-self-dual spin connection and are overtly invariant under anti-self-dual $SL(2, \mathbb{C})$ gauge transformations. The self-dual spin connection and gauge transformations could equally well have been used. In the real Lorentzian case, for example, taking the complex conjugate effects this change. However, it is the case that the Lagrangians for the self-dual formalisms can be expressed directly in terms of the geometrical objects and gauge transformations introduced earlier. This is most simply demonstrated by introducing a chiral four-spinor notation in which the components of the chiral four-spinor-valued 1-form (α^A, β^A) and the matrix-valued connection forms

$$\begin{bmatrix} \omega^A_B & 0 \\ 0 & \omega^A_B \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} \pi_1 1 & \pi_2 1 \\ \pi_3 1 & -\pi_1 1 \end{bmatrix} \tag{50}$$

are respectively denoted by χ^i, Γ^i_j, A^i_j. Here 0 and 1 denote the zero and unit 2×2 matrices, and Latin indices range over 1 to 4. Using this notation, we find that the vacuum equations represented by equations (2) and (7) and equation (3) are equivalent to the equations

$$d\chi^i = \chi^j \wedge \Gamma^i_j + \chi^j \wedge A^i_j \tag{51}$$

and

$$\chi^j \wedge (dA^i_j + A^i_k \wedge A^k_j) = \chi^j \wedge (d\Gamma^i_j + \Gamma^i_k \wedge \Gamma^k_j) = 0 \tag{52}$$

The metric given by equation (6) takes the form

$$ds^2 = g_{ij}\chi^i \otimes \chi^j \tag{53}$$

where

$$g_{ij} = \begin{bmatrix} 0 & \varepsilon_{AB} \\ -\varepsilon_{AB} & 0 \end{bmatrix} \tag{54}$$

and $\Gamma_{ij} = -\Gamma_{ji}$, $A_{ij} = -A_{ji}$.

As an illustration, consider the Lagrangian of Tung and Jacobson given in equation (5). It can now be written as

$$L_1 = \frac{1}{2} D\chi^i \wedge D\chi_i \tag{55}$$

where D is the covariant exterior derivative determined by Γ^i_j. The Lagrangian L_1 is invariant under $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ transformations, where $SL(2, \mathbb{C})$ is the gauge group of anti-self-dual transformations. The Lagrangian

$$L_4 = \frac{1}{2} \nabla\chi^i \wedge \nabla\chi_i \tag{56}$$

is also a Lagrangian for the vacuum equations, but now the exterior covariant derivative ∇ is determined by (the self-dual) A^i_j and the gauge group corresponds to the self-dual $SL(2, \mathbb{C})$ transformations as in equation (36).

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